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Prime ideals in the algebra of holomorphic functions of several complex variables

by

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1. Introduction

Let Ω be a domain of holomorphy in the complex Euclidean space $\mathbb{C}^N$, and Θ be the algebra of holomorphic functions in Ω, with the topology of uniform convergence on compact subsets of Ω.

We studied maximal ideals in Θ [3], and also prime ideals in Θ, for the one-variable case, in some detail [4]. For the several-variables case, we treated only those prime ideals which satisfy the following condition:

An ideal J in Θ is said to satisfy the (z)-*condition*, if a function $f\in\Theta$ belongs to J whenever there is a finite set of functions $g_1, \cdots, g_r \in J$ such that

$$\{f=0\}\supset\{g_1=\cdots=g_r=0\}.$$

Of course, a maximal ideal is a prime (z)-ideal.

Now we will set about to study prime ideals for the several-variables case. Our results in this note are yet very restricted and meager ones, but we expect that they would provide a starting point for further studies.

2. Preliminaries

1° Let J be an ideal in Θ. Then, we can choose a finite set of functions $f_1, \cdots, f_t \in J$ with the following property:

We denote by $\{A_n\}$ the set of all irreducible components of $\{f_1=\cdots=f_t=0\}$. Let $g_1, \cdots, g_r \in J$ be an arbitrary finite set of functions in J and let $\{A_{nk}\}$, $A_{nk}\subset A_n$, be the set of all irreducible components of $\{f_1=\cdots=f_t=g_1=\cdots=g_r=0\}$. Then there are numbers n and k such that $A_{nk}=A_n$. (For the proof, see [5, Theorem 1].)

We call this $(f_1, \cdots, f_t)$ as an (F)-*system*, and $Z=\{A_n\}$ (the set of all irreducible components of $\{f_1=\cdots=f_t=0\}$) as an (FA)-*set*, for the ideal J.

By the way, we write $(h_1, \cdots, h_m)$, $h_j\in\Theta$, simply as $(h)_m$, and $\{h_1=\cdots=h_m=0\}$ as $\{(h)_m=0\}$.

Let I be the set of all positive integers. A family Φ of subsets of I is defined as follows: $E\subset I$ belongs to Φ if there is a $(g)_r\subset J$ such that

$$E=\{k\in I; \{(f)_t=(g)_r=0\}\supset A_k\} .$$

$((f)_t$ is the (F)-system for J.) Then, Φ is easily seen to be a filter, see [5].

2° If J is a maximal ideal, then the (FA)-set $\{A_n\}$ is a point sequence [5]. If J is a prime (z)-ideal (see Introduction), then the filter Φ is a ultrafilter [5]. Thus, a prime (z)-ideal is characterized by a pair (Z, Φ) of a sequence $Z=\{A_n\}$ of irreducible components of an analytic set and a ultrafilter Φ on the set of positive integers I.

3° For an ideal J in Θ, we put

$$S^z(J)=\bigcap \{K; K \text{ is a } (z)\text{-ideal containing } J\} .$$

Then, $S^z(J)$ is the smallest (z)-ideal containing J.

4° Let A be an irreducible analytic set in Ω. We denote by $I(A)$ the ideal of A, i.e., the totality of holomorphic functions in Θ which vanish on A. For a positive integer p, we put

$$I_p(A)=\{f\in\Theta; f \text{ belongs to } I(A)^p \text{ on every analytic polyhedron in } \Omega\} ,$$

and $I_0(A)=\Theta$. $I_p(A)$, $p\geqq 0$, is easily seen to be an ideal which is closed (for the definition of analytic polyhedron, see [1, p. 52, *domaine polyédral*] or [2, p. 45 or p. 216]). We do not know whether $I_p(A)$ would coincide with $I(A)^p$ or not.

LEMMA 2.1. (i) $I_{p+1}(A)\subsetneqq I_p(A)$.
(ii) $\bigcap_p I_p(A)=\{0\}$.

The *proof* is easy and may be omitted.

5° For $(g)_r\subset\Theta$, let $I((g)_r)$ be the ideal generated by $(g)_r$. By Lemma 2.1 (ii), there is an integer $p\geqq 0$ such that

$$I((g)_r)\subset I_p(A) , \quad I((g)_r)\not\subset I_{p+1}(A) .$$

We call this p the *multiplicity* of $I((g)_r)$ (or, simply, of $(g)_r$) at A.

If $r=1$, then $(g)_r$ is written simply as g.

6° For $(g)_r\subset\Theta$, let $\{B_n\}$ be the set of all irreducible components of $\{(g)_r=0\}$ and p_n be the multiplicity of $(g)_r$ at B_n. Put

$$K=K((g)_r)=\bigcap_n I_{p_n}(B_n) .$$

Then, $K\supset I((g)_r)$.

We define an ideal J is said to satisfy the *(zm)-condition* if

$$(g)_r\subset J \quad \text{implies} \quad K((g)_r)\subset J .$$

If we denote by $S^{zm}(J)$ the smallest (zm)-ideal containing J, it is easy to see that the (FA)-set for $S^{zm}(J)$ is the (FA)-set for J also and vice versa.

7° Let A be an irreducible analytic set in Ω and $p(f)$ be the multiplicity of a function f at A. Then, we have

$$p(fg)=p(f)+p(g) \qquad \text{for } f, g \in \Theta . \tag{2.1}$$

Proof. If f belongs to $I(A)^p$ at any point $x \in A$, then f belongs to $I_p(A)$. Hence, if the multiplicity of f at A is p, then there is a point $x \in A$ such that f belongs to $I(A)^p - I(A)^{p+1}$ at x. The set

$$\{x \in A; f \text{ belongs to } I(A)^{p+1} \text{ at } x\}$$

is obviously an analytic set of dimension less than $\dim(A)$. Therefore

$$\{x \in A; f \text{ belongs to } I(A)^p - I(A)^{p+1} \text{ at } x\}$$

is open and dense on A.

Thus, there is an ordinary point x of A such that

$$h \text{ belongs to } I(A)^{p(h)} - I(A)^{p(h)+1} \text{ at } x ,$$

where h denotes any one of fg, f, and g. Since locally we have obviously

$$p(fg)=p(f)+p(g) ,$$

we obtain (2.1).

8° We conjecture that a prime ideal P would satisfy the (zm)-condition, though we can not prove it.

In this note, we treat only those prime ideals which satisfy the (zm)-condition.

3. Characterizations of prime (zm)-ideals

Let P be a prime (zm)-ideal, with $(f)_t$ as an (F)-system and $Z=\{A_n\}$ as an (FA)-set. For the set I of all positive integers, a filter Φ is induced by the ideal P, as stated in § 2, 1°. We denote by p_n the multiplicity of $(f)_t$ at A_n (see § 2, 5°).

Let E be a subset of I. Set $K=K((f)_t)$ and

$$L=\bigcap_{n\in E} I_{p_n}(A_n) , \quad H=\bigcap_{n\in I-E} I_{p_n}(A_n) .$$

K is contained in P by the assumption of the (zm)-condition for P. Since

$$HL \subset H \cap L = K \subset P ,$$

we have

$$H \subset P \quad \text{or} \quad L \subset P .$$

Suppose $L \subset P$. Take $z_k \in A_k$, $k \in I-E$, such that $z_k \notin \bigcup_{n\in E} A_n$. We can see easily that there is a function $g \in L$ with $g(z_k) \neq 0$ for all $k \in I-E$. Then $g \in P$, which shows that $E \in \Phi$.

If we suppose that $H \subset P$, then we have similarly $I-E \in \Phi$.

Thus, we know that Φ is a ultrafilter. It follows that the smallest (z)-ideal $Q=S^Z(P)$ containing P (see § 2, 3°) is a prime ideal.

Thus, every prime (zm)-ideal P has a *prime* (z)-ideal $Q=(Z, \Phi)$ ($Z=\{A_n\}=$the sequence of all irreducible components of an analytic set, Φ = a ultrafilter on I) as the smallest (z)-ideal $S^Z(P)$.

In the converse direction, we will consider the following problem: Suppose a prime (z)-ideal $Q=(Z, \Phi)$ is given, where $Z=\{A_n\}$ is a sequence of all irreducible components of an analytic set, and Φ is a ultrafilter on the set I of all positive integers. Are there prime (zm)-ideal P such that P is not (z)-ideal and $S^Z(P)=Q$? And, if there exist such prime (zm)-ideals, how we can characterize them?

Now, let (Z, Φ) be given. For a $(g)_r \subset \Theta$, we denote by $p_n((g)_r)$ the multiplicity of $(g)_r$ at A_n.

Let $\phi(t)$ be a positive function of t, $0<t<\infty$. We denote by $J(\phi)$ the set of functions in Θ defined as follows: f belongs to $J(\phi)$ if there are a positive number $\delta=\delta(f)$ and a set $E=E(f) \in \Phi$ such that

$$p_n(f) \geqq \delta\phi(n) \qquad \text{if } n \in E\,.$$

$J(\phi)$ is obviously an ideal. We will show that $J(\phi)$ is prime.

Suppose $fg \in J(\phi)$. Then, by § 2, 7°,

$$p_n(fg)=p_n(f)+p_n(g) \geqq \delta\phi(n) \quad \text{for } n \in E$$

with some $E \in \Phi$ and $\delta>0$. Put

$$E_1=\{n \in I;\, p_n(f) \geqq (\delta/2)\phi(n)\}\,.$$

If $E_1 \in \Phi$, then $f \in J(\phi)$. If $E_1 \notin \Phi$, then $I-E_1$, hence $(I-E)\cap E_1$, belongs to Φ, which shows that $g \in J(\phi)$. Thus $J(\phi)$ is a prime ideal.

Of course, $J(\phi)$ is a (zm)-ideal.

$J(\phi)$ is a (z)-ideal if and only if $\phi(t)$ is bounded in the sense that there are a set $E \in \Phi$ and a constant B such that

$$\phi(n) \leqq B \quad \text{for } n \in E\,. \tag{3.1}$$

Let ϕ_1 and ϕ_2 be positive functions of $t>0$. We write $\phi_1 \leqq \phi_2$ if there are a set $E \in \Phi$ and a constant $\delta>0$ such that

$$\phi_1(n) \geqq \delta\phi_2(n) \quad \text{for } n \in E\,. \tag{3.2}$$

If $\phi_1 \leqq \phi_2$ and $\phi_2 \leqq \phi_1$, then we write $\phi_1 \cong \phi_2$. Then, the set of all positive function $\phi(t)$, $t>0$, is easily seen to be *totally ordered.*

If $\phi_1 \lneqq \phi_2$, we have $\phi_1 \lneqq \sqrt{\phi_1\phi_2} \lneqq \phi_2$.

Sometimes, $\phi_1 \lneqq \phi_2$ is written simply as $\phi_1 \prec \phi_2$.

THEOREM 3.1. *$J(\phi_1)\subseteqq J(\phi_2)$ if and only if $\phi_1\leqq\phi_2$.*

Proof. Clearly (3.2) implies that $J(\phi_1)\subseteqq J(\phi_2)$.

Conversely, suppose $J(\phi_1)\subseteqq J(\phi_2)$. We may assume that ϕ_1 is not bounded in the sense of (3.1). Hence, there is a set $E_1\in\Phi$ such that

$$\phi_1(k)\geqq 2 \qquad \text{if } k\in E_1 .$$

Let $f(z)$ be a function such that

$$\begin{aligned} p_k(f)&=[\phi_1(k)]+1 && \text{if } k\in E_1 ,\\ &=0 && \text{otherwise} ,\end{aligned} \tag{3.3}$$

where $[x]$ denotes the largest integer not greater than $x>0$. Such a function exists, as shown presently. Then, f belongs to $J(\phi_1)$, hence $f\in J(\phi_2)$. Thus, there are a set $E_2\in\Phi$ and a constant $\delta_2>0$ such that

$$p_k(f)\geqq\delta_2\phi_2(k) \qquad \text{if } k\in E_2 .$$

Thus, if $E=E_1\cap E_2$ and $k\in E$,

$$\phi_1(k)\geqq(\phi_1(k)+2)/2\geqq([\phi_1(k)]+1)/2=p_k(f)/2\geqq\delta\phi_2(k)$$

with $\delta=\delta_2/2$, which proves (3.2).

Existence of a function satisfying (3.3) is implied in

LEMMA. *Let $\{A_n\}$ be a sequence of irreducible components of an analytic set, and $\{p_n\}$ be a sequence of non-negative integers. Then, there is a function $f\in\Theta$ which has the multiplicity p_n at A_n for each n.*

Proof. For each n, take a function $g_n\in\Theta$ such that

$$g_n\in\bigcap_{m\neq n} I_{p_m+1}(A_m)$$

and $g_n(z_n)\neq 0$ for a point $z_n\in A_n$. Further, take a function $h_n\in\Theta$ such that

$$h_n\in I(A_n)-I(A_n)^2 .$$

Let $\{K_n\}$ be a sequence of closed subdomains of Ω such that

K_n is contained in the interior of K_{n+1}, and

$$\bigcup_n K_n=\Omega .$$

Put

$$f_n(z)=g_n(z)(h_n(z))^{p_n}$$

and

$$R_n=\max_{z\in K_n}|f_n(z)| .$$

Then, the function

$$f(z)=\sum 2^{-n}R_n^{-1}f_n(z)$$

is easily seen to have the desired properties. Q.E.D.

Let P be a prime (zm)-ideal with a pair (Z, Φ). For any $f\in\Theta$, we define a positive function ϕ_f to be

$$\phi_f(k)=\max(p_k(f), 1) \qquad \text{for } A_k\in Z .$$

Then, we have $J(\phi_f)\subseteqq P$ if $f\in P$. Further, there holds that if $J(\phi)\not\subseteqq P$, then $J(\phi)\supset P$.

For, take $f\in J(\phi)-P$. We can see without difficulty that, for any function $g\in P$,

$$\phi_g \lesssim \phi_f ,$$

hence we have $P\subseteqq J(\phi_f)\subseteqq J(\phi)$.

We have immediately

THEOREM 3.2. *If P is a prime (zm)-ideal, then*

$$P=\bigcup\{J(\phi);\ J(\phi)\subseteqq P\} .$$

Now we denote by W the set of all positive functions $\phi(t)$, $t>0$. W is totally ordered by the relatition (3.2). We define "cut" of W, in analogy to the Dedekind cut of the set of rational numbers:

DEFINITION. A subset $\mathfrak{A}$ of W is said to be a *cut* of W if

(i) $\mathfrak{A}$ is a non-void proper subset of W,

(ii) If $\phi\in\mathfrak{A}$ and $\psi\lesssim\phi$, then $\psi\in\mathfrak{A}$.

We denote by Υ the set of all cuts of W. Υ can be totally ordered as usual.

For $\phi_1, \phi_2\in W$, we define $\phi_1\phi_2\in W$ to be $(\phi_1\phi_2)(t)=\phi_1(t)\phi_2(t)$. Then, the multiplication is introduced in Υ as usual, and Υ becomes an abelian group.

We endow Υ with the order topology, i.e., a fundamental system of neighborhoods of $\mathfrak{A}\in\Upsilon$ consists of subsets of the form (ϕ_1, ϕ_2) with $\phi_1, \phi_2\in W$, $\phi_1\prec\mathfrak{A}\prec\phi_2$, where

$$(\phi_1, \phi_2)=\{\mathfrak{A}'\in\Upsilon;\ \phi_1\prec\mathfrak{A}'\prec\phi_2\} .$$

Υ is easily seen to be a locally compact topological group, which is abelian and totally ordered.

For every cut $\mathfrak{A}\in\Upsilon$, there corresponds a prime (zm)-ideal $J(\mathfrak{A})$ uniquely, defined by

$$J(\mathfrak{A})=\bigcup_{\phi\in\mathfrak{A}} J(\phi) . \tag{3.4}$$

Conversely, theorem 3.2 states that every prime (zm)-ideal determines a cut $\mathfrak{A}\in\Upsilon$ in a unique way.

$J(\mathfrak{A})$ is a (z)-ideal if and only if $\mathfrak{A}$ contains a bounded function in the sense of (3.1).

As easily seen, the unity in the abelian group W is represented by a function ϕ_0 such that there are a set $E \in \Phi$ and constants K_1, K_2, $K_1 > K_2 > 0$ such that

$$K_2 \leqq \phi_0(k) \leqq K_1 \qquad \text{if } k \in E .$$

The (z)-ideal can be written as $J(\phi_0)$.

We define a product P_1P_2 of two prime (zm)-ideals P_1 and P_2 as follows: If $P_1 = J(\mathfrak{A}_1)$ and $P_2 = J(\mathfrak{A}_2)$, then

$$P_1P_2 = J(\mathfrak{A}_1\mathfrak{A}_2) .$$

Thus, if a (z)-ideal $Q = (Z, \Phi)$ is given, then the set S of prime (zm)-ideals

$$S = \{P; S^z(P) = Q\} \quad (S^z(P) \text{ is defined in } \S\, 2,\ 3^\circ)$$

can be considered as a (totally ordered) semi-group which is locally compact and abelian. We denote this semi-group as $\sum = \sum(\Phi)$. The operator J defined by (3.4) is a homomorphism of Υ onto $\sum$, which preserves the order relation.

References

[1] Cartan, H.; Idéaux et Modules de Fonctions Analytiques de Variables Complexes, *Bull. Soc. Math. France*, 78 (1950), 28–64.

[2] Gunning, R. C. and Rossi, H.; *Analytic Functions of Several Complex Variables*. Prentice-Hall Inc., Englewood Cliffs, N.J,, 1965.

[3] Yanagihara, N.; On a theorem of Igusa II, to appear.

[4] Yanagihara, N.; Prime ideals in the algebra of holomorphic functions, to appear.

[5] Yanagihara, N.; Prime ideals and maximal ideals in the algebra of holomorphic functions, to appear.

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